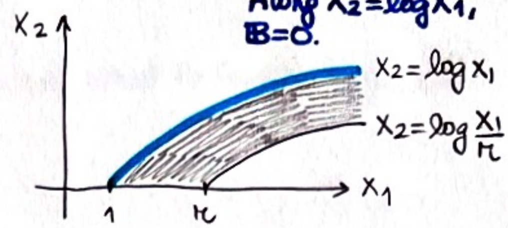
4. Boundary Condition: $\boxed{\mathbb{B}(x_1, \log x_1) = 0}$



Take $x \in \Omega_r$ with $x_2 = \log x_1$.
This means equality in Jensen though,
$$\langle \log w \rangle_J = \log \langle w \rangle_J$$
which occurs only for constant functions. Since $x_1 = \langle w \rangle_J$,

The only admissible weight for $\mathbb{B}(x_1, \log x_1)$ is the constant weight $w \equiv x_1$ a.e.

But clearly $\mathbb{B} = 0$ in this case, since the quantity in the supremum is 0 for constant w.

5. Homogeneity: Suppose $w \in A_\infty^{\partial}(J; \kappa)$ is admissible for $\mathbb{B}(x)$, for some $x \in \Omega_r$, i.e.
$$\langle w \rangle_J = x_1; \quad \langle \log w \rangle_J = x_2.$$
Let $\tilde{w} := \tau w$ be a new weight, for some $\tau > 0$.
$$\langle \tilde{w} \rangle_I = \tau \langle w \rangle_I \le \tau \kappa e^{\langle \log w \rangle_I} = \kappa e^{\log \tau + \langle \log w \rangle_I} = \kappa e^{\langle \log \tilde{w} \rangle_I} \Rightarrow \tilde{w} \in A_\infty^{\partial}(J; \kappa)$$
$$\left.\begin{aligned} \langle \tilde{w} \rangle_J &= \tau x_1 \\ \langle \log \tilde{w} \rangle_J &= \log \tau + x_2 \end{aligned}\right\} \Rightarrow \tilde{w} \text{ is admissible for } \mathbb{B}(\tau x_1, \log \tau + x_2)$$
The expression in the supremum does not see the τ (homogeneous of degree 0 wrt w)
$$\frac{\langle \tilde{w} \rangle_{I_+} - \langle \tilde{w} \rangle_{I_-}}{\langle \tilde{w} \rangle_I} = \frac{\tau \langle w \rangle_{I_+} - \tau \langle w \rangle_{I_-}}{\tau \langle w \rangle_I} = \frac{\langle w \rangle_{I_+} - \langle w \rangle_{I_-}}{\langle w \rangle_I}$$
so:
$$\boxed{\mathbb{B}(x_1, x_2) = \mathbb{B}(\tau x_1, \log \tau + x_2), \quad \forall \tau > 0}$$
Take a convenient value of τ above: $\tau = e^{-x_2}$ (in essence, this eliminates a variable):
$$\mathbb{B}(x_1, x_2) = \mathbb{B}(e^{-x_2} x_1, 0) =: g(x_1 e^{-x_2}) =: g(s)$$
$$\Rightarrow \boxed{\begin{aligned} &\mathbb{B}(x_1, x_2) = g(s), \text{ where} \\ &s := x_1 e^{-x_2}, \; s \in [1, \kappa] \end{aligned}}$$
Remark: In this case the Boundary condition
$$0 = \mathbb{B}(x_1, \log x_1) = g(x_1 e^{-\log x_1}) = g(1)$$
becomes $\boxed{g(1) = 0}$

$$\log \frac{x_1}{\kappa} \le x_2 \le \log x_1$$
$$\frac{x_1}{\kappa} \le e^{x_2} \le x_1$$
$$\frac{\kappa}{x_1} \ge e^{-x_2} \ge \frac{1}{x_1}$$
$$\kappa \ge x_1 e^{-x_2} \ge 1 \; //$$

Remark: Let us reframe the previous Theorem 1 in terms of what it really says about $\mathbb{B}$:

6. Least Supersolution:

We define a <u>supersolution</u>* for this problem to be any positive function $B(x_1, x_2)$ that satisfies the Main Inequality.

[*) Most Bellman problems also have an <u>Obstacle Condition</u>, and supersolutions are usually functions that satisfy both the MI and the OC.]

So the Bellman induction proof in Theorem 1 really showed that: if B is any supersolution, then

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(\frac{\langle w\rangle_{I+}-\langle w\rangle_{I-}}{\langle w\rangle_I}\right)^2|I| \leq B(x), \ \forall\, w\in A_\infty^{\mathcal{D}}(J;x)\ \text{w/}\ \langle w\rangle_J = x_1 \ \& \ \langle \log w\rangle_J = x_2$$

But then obviously

$$\mathbb{B}(x) \leq B(x) \ !!$$

If $B(x)$ is any supersolution, then $\mathbb{B}(x) \leq B(x)$.
In other words: THE "TRUE" BELLMAN FUNCTION $\mathbb{B}$ IS THE <u>LEAST SUPERSOLUTION</u>.

This has a major significance, namely: **in reality, to prove this inequality, we don't actually have to find the true Bellman function $\mathbb{B}$; it is enough to find <u>ANY</u> supersolution !!!**

The "true" Bellman functions are known for very few select problems. Most results proved via the Bellman function method actually use only a supersolution, found through clever guesses, experience, or actual black magic (whatever works!) :)

NEXT STEP: How to find a supersolution ?!

Differential Form of Main Inequality: Here B is any function satisfying the Main Inequality, and assumed to be twice differentiable.

(Making smoothness assumptions is OK - in a quest to find a supersolution, we can make any assumptions we want!)

Discrete MI: $\quad B(x) \geq \frac{1}{2}\left(B(x_+)+B(x_-)\right)+\left(\frac{x_1^+ - x_1^-}{x_1}\right)^2$

Recall: Multivariate Taylor Formula: for an open subset $U \subset \mathbb{R}^n$, a function $f: U \to \mathbb{R}$ w/ continuous 2^{nd} order partial derivatives on U, as y approaches x:

$$f(y) = f(x) + \langle \nabla f(x), y-x \rangle + \frac{1}{2}(y-x)^T (Hf(x))(y-x) + o(\|y-x\|^2)$$

means $\lim_{\|y-x\|\to 0} \frac{o(\|y-x\|^2)}{\|y-x\|^2} = 0$

Take $x \in \Omega_r$, a small $t>0$, and $(\xi_1, \xi_2) \in \mathbb{R}^2$.
Apply the MI to $\quad \xi_t := (t\xi_1, t\xi_2)$

$$x^- = x - (t\xi_1, t\xi_2)$$
$$x^+ = x + (t\xi_1, t\xi_2)$$

(Assuming $x \in \Omega_r^\circ$, $x^-, x^+ \in \Omega_r$ for small enough t)

$$B(x+\xi_t) = B(x) + \langle \nabla B(x), \xi_t \rangle + \frac{1}{2}\xi_t^T (HB(x))\xi_t + o(\|\xi_t\|^2)$$
$$B(x-\xi_t) = B(x) - \langle \nabla B(x), \xi_t \rangle + \frac{1}{2}\xi_t^T (HB(x))\xi_t + o(\|\xi_t\|^2)$$

$\oplus$ $\quad B(x^+) + B(x^-) = 2B(x) + \xi_t^T (HB(x))\xi_t + 2o(\|\xi_t\|^2)$

$$\Rightarrow 2B(x) \geq 2B(x) + \left(\xi_1^2 B_{11} + 2\xi_1\xi_2 B_{12} + \xi_2^2 B_{22}\right)t^2 + 2o\left(t^2(\xi_1^2+\xi_2^2)\right) + \left(\frac{2t\xi_1}{x_1}\right)^2 \cdot 2$$

$$\Rightarrow 0 \geq \left[\xi_1^2\left(B_{11} + \frac{8}{x_1^2}\right) + 2\xi_1\xi_2 B_{12} + \xi_2^2 B_{22}\right]t^2 + 2o\left(t^2(\xi_1^2+\xi_2^2)\right) \quad \Big| \cdot \frac{1}{t^2(\xi_1^2+\xi_2^2)}$$

$\lim_{t\to 0}$

$$\Rightarrow 0 \geq \xi^T \begin{pmatrix} B_{11} + \frac{8}{x_1^2} & B_{12} \\ B_{12} & B_{22} \end{pmatrix} \xi \quad \forall\, \xi \in \mathbb{R}^2$$

$\Rightarrow$ **The 2×2 matrix is negative semidefinite!** $\quad \begin{pmatrix} B_{11} + \frac{8}{x_1^2} & B_{12} \\ B_{12} & B_{22} \end{pmatrix} \leq 0$

But recall that our function satisfies $B(x_1,x_2)=g(s)$, $s=x_1e^{-x_2}$, $s\in[1,r]$.
So impose
$$B(x_1,x_2)=g(s),\quad s=x_1e^{-x_2}$$
and express the matrix
$$\begin{pmatrix} B_{11}+\frac{8}{x_1^2} & B_{12} \\ B_{12} & B_{22}\end{pmatrix}$$
in terms of $g(s)$:
$$B_1=\frac{\partial B}{\partial x_1}=g'(s)\cdot e^{-x_2}$$
$$\Rightarrow B_{11}=g''(s)\cdot e^{-x_2}e^{-x_2}=g''(s)\cdot e^{-2x_2}$$
$$\Rightarrow B_{11}+\frac{8}{x_1^2}=g''(s)\cdot e^{-2x_2}+\frac{8}{x_1^2}$$
$$=e^{-2x_2}\left(g''(s)+\frac{8}{(x_1e^{-x_2})^2}\right)=\boxed{e^{-2x_2}\left(g''(s)+\frac{8}{s^2}\right)}$$
$$\Rightarrow B_{12}=g''(s)\cdot(-x_1e^{-x_2})e^{-x_2}-g'(s)\cdot e^{-x_2}$$
$$=-e^{-x_2}\left(g''(s)\cdot s+g'(s)\right)=\boxed{-e^{-x_2}(g'(s)\cdot s)'}$$

$$B_2=g'(s)\left(-x_1e^{-x_2}\right)=-g'(s)x_1e^{-x_2}$$
$$\Rightarrow B_{22}=+g''(s)\cdot x_1^2e^{-2x_2}+g'(s)x_1e^{-x_2}$$
$$=\boxed{g''(s)\cdot s^2+g'(s)\cdot s}=\boxed{s\cdot(g'(s)\cdot s)'}$$

$\Rightarrow$ New Matrix condition:
$$\begin{pmatrix} e^{-2x_2}\left(g''+\frac{8}{s^2}\right) & -e^{-x_2}(g'\cdot s)' \\ -e^{-x_2}(g'\cdot s)' & s\cdot(g'(s)\cdot s)'\end{pmatrix}\leq 0$$

Equivalent to scalar inequalities:
$$\begin{cases} g''+\frac{8}{s^2}\leq 0 \\ (sg')'\leq 0\end{cases}$$
$\oplus$ the determinant condition $\det\geq 0$.

Take the determinant to be 0:

$$e^{-2X_2}\left(g''+\frac{8}{s^2}\right)s(sg')' - e^{-2X_2}\left((sg')'\right)^2 = 0$$

$$\left(sg''+\frac{8}{s}-(sg')'\right)(sg')'=0$$

$$\left(\cancel{sg''}+\frac{8}{s}-\cancel{sg''}-g'\right)(sg')'=0$$

$$\boxed{\left(g'-\frac{8}{s}\right)(sg')'=0}$$

$g'=\frac{8}{s} \Rightarrow g = 8\log s + c_1$ | $(sg')'=0 \Rightarrow sg'=c \Rightarrow g'=\frac{c}{s}$

$\Rightarrow g = c\log s + c_1$

$\Rightarrow$ General Solution: $g(s) = c\log s + c_1$

Boundary condition: $g(1)=0 \Rightarrow c_1 = 0$

How to choose c? Go back to:

$$\begin{cases} g''+\frac{8}{s^2} \leq 0 \\ (sg')' \leq 0 \end{cases}$$

$g = c\log s \Rightarrow g' = \frac{c}{s} \Rightarrow g'' = \frac{-c}{s^2}$

$$\begin{cases} -\frac{c}{s^2}+\frac{8}{s^2} \leq 0 \\ 0 \leq 0 \end{cases} \Rightarrow c \geq 8$$

Since we want g to be as small as possible, choose $c=8$.

$$g(s) = 8\log s \Rightarrow B(X_1,X_2) = 8\log(X_1 e^{-X_2}) = 8(\log X_1 - X_2)$$

$\Rightarrow B(X_1,X_2) = 8(\log X_1 - X_2)$ is a supersolution candidate for the problem, i.e. now we must show that B satisfies the discrete MI (already proved).